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The Meijer's G-functions Convenient for Describing β & γ-decays

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Abstract: The purpose of this paper is to show that the Meijer's G-functions, as a very general complex functions which include all elementary, and most of the special functions, can describe some phenomena in nuclear physics. In fact, some interesting properties of Meijer's G-functions triggered us to apply these functions in the processes β and γ-decay. The role of these functions is as "wave function" for parent and daughter nuclei.

Keywords: Meijer's G-function, β decay; γ-decay, Special functions, Gamma function.

1. INTRODUCTION

In the recent decades, Meijer's G-functions (*MGFs*) has found various applications in different areas close to applied mathematics, such as mathematical physics, theoretical physics, mathematical statistics etc. Due to the important properties of the *MGFs,* it is possible to represent the solutions of many problems in terms of *MGFs* [1]. Recently Pishkoo and Darus obtain G-function solutions for Reaction-diffusion equation [2], Schrödinger equation [3], Diffusion equation, and Laplace's equation [4, 5], respectively.

Stated in this way, the problems gain a general character, due to the great freedom of choice of orders and parameters of *MGFs,* in comparison to the other special functions. Simultaneously, the calculations have become simpler and unified. An evidence showing the importance of *MGFs* is given by the fact that the basic elementary functions and most of the special functions of Mathematical Physics, including the generalized hypergeometric functions, follow as its particular cases (see e.g. [1]).

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Table 1: Derivation of excited radial states of the hydrogen atom from the ground state R_{10} in terms of Meijer's G-function $G_{0,1}^{1,0}$

Radial function	Meijer's G-function
R_{10}	$G_{0,1}^{1,0}\left(\begin{matrix}-\\0\end{matrix}\middle\vert\frac{r}{r_B}\right)$
R_{21}	$G_{0,1}^{1,0}\left(\begin{matrix}-\\1\end{matrix}\middle\vert\frac{r}{2r_B}\right)$
R_{20}	$G_{1,2}^{1,1}\left(\begin{matrix}0\\0,1\end{matrix}\middle\vert\frac{r}{2r_B}\right)=G_{0,1}^{1,0}\left(\begin{matrix}-\\0\end{matrix}\middle\vert\frac{r}{2r_B}\right)+G_{0,1}^{1,0}\left(\begin{matrix}-\\1\end{matrix}\middle\vert\frac{r}{2r_B}\right)$
R_{32}	$G_{0,1}^{1,0}\left(\begin{matrix}-\\2\end{matrix}\middle\vert\frac{r}{3r_B}\right)$
R_{31}	$G_{0,1}^{1,0}\left(\begin{matrix}-\\1\end{matrix}\middle\vert\frac{r}{3r_B}\right)+G_{0,1}^{1,0}\left(\begin{matrix}-\\2\end{matrix}\middle\vert\frac{r}{3r_B}\right)$
R_{30}	$G_{0,1}^{1,0}\left(\begin{matrix}-\\0\end{matrix}\middle\vert\frac{r}{3r_B}\right)+G_{0,1}^{1,0}\left(\begin{matrix}-\\1\end{matrix}\middle\vert\frac{r}{3r_B}\right)+G_{0,1}^{1,0}\left(\begin{matrix}-\\2\end{matrix}\middle\vert\frac{r}{3r_B}\right)$

In [6] we have applied some properties of Meijer's *G*-function for obtaining excited states (radial functions) of the Hydrogen atom from ground state. Our results are summarized in the following table. As it is shown in Table 1, all radial states of Hydrogen atom belong to the family of functions $G_{0,1}^{1,0}$ which in their combinations the value of parameter is different.

All of these states are shown by Meijer's *G*-function. Because many obtaining results from atomic physics can be generalized to nuclear physics, here we assume that the wave functions of parent and daughter nuclei are of type Meijer's G-function, and then we describe β & γ-decays.

2. MEIJER'S *G*-FUNCTIONS

We begin with the definition of Meijer's *G*-function as the following:

Definition 2.1 *A definition of the Meijer's G-function is given by the following path integral in the complex plane, called Mellin-Barnes type integral [1, 7, 8, 9, 10]:*

$$G_{p,q}^{m,n}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle| z\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}z^s ds. \tag{2.1}$$

Here, the integers $m;n;p;q$ are called "orders" of the G-function, or the components of the order $(m;n;p;g)$; a_j and b_j are called "parameters" and in general, they are complex numbers. The definition holds under the following assumptions: $0\le m\le q$ and $0\le n\le p$ where m;n,p, and q are integer numbers. Subtracting parameters $a_j-b_k\ne 1,2,3,\ldots$ for k = 1, ... ,n and j =1,2,...,m imply that no pole of any $\Gamma(b_j-s), j=1,\ldots,m$ coincides with any pole of any $\Gamma(1-a_k+s), k=1,\ldots,n$

Choosing m =1, n = 0, p = 0 and q = 1, in the simplest case we have

$$G_{0,1}^{1,0}\left(\begin{matrix}-\\ b_1\end{matrix}\middle| z\right)=\frac{1}{2\pi i}\int_L \Gamma(b_1-s)z^s ds. \tag{2.2}$$

Based on the definition, the following basic properties are easily derived:

$$z^{\alpha}G_{p,q}^{m,n}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)=G_{p,q}^{m,n}\left(\begin{matrix}\mathrm{a_p}+\alpha\\ \mathrm{b_q}+\alpha\end{matrix}\middle| z\right), \tag{2.3}$$

where the multiplying term z^{α} changes the parameters of the G-function. The derivatives of arbitrary order k can change the G-function's orders and parameters:

$$z^k\frac{d^k}{dz^k}G_{p,q}^{m,n}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)=G_{p+1,q+1}^{m,n+1}\left(\begin{matrix}0,\mathrm{a_p}\\ \mathrm{p_q},\mathrm{k}\end{matrix}\middle| z\right). \tag{2.4}$$

The G-function is symmetric in the groups of parameters $(a_1,\ldots,a_n),(a_{n+1},\ldots,a_p)$, $(b_1,\ldots,b_m),(b_{m+1},\ldots,b_q)$. If one of the a_j's, j = 1,..., n, is equal to some of the b_k's, k = m +1, ..., q, then the G-function reduces to one of Lower order. For example, if $n,p,q\ge 1$

$$G_{p,q}^{m,n}\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_{q-1},\mathrm{a}_1\end{matrix}\middle| z\right)=G_{p-1,q-1}^{m,n-1}\left(\begin{matrix}a_2,\ldots,a_p\\ b_1,\ldots,b_{q-1}\end{matrix}\middle| z\right). \tag{2.5}$$

Similarly, if one of the aj's j=n+1,..., p, is equal to some of the b_k, k = 1, ..., m, the function has its order m,p,q reduced by 1, for example, if $m,p,q\ge 1$,

$$G_{p,q}^{m,n}\left(\begin{matrix}a_1,\ldots,a_{p-1},b_1\\ b_1,b_2,\ldots,b_q\end{matrix}\middle| z\right)=G_{p-1,q-1}^{m-1,n}\left(\begin{matrix}a_1,\ldots,a_{p-1}\\ b_2,\ldots,b_q\end{matrix}\middle| z\right). \tag{2.6}$$

3. BETA DECAY

In β-decay the charge of a nucleus changes while mass remains fixed. This process occurs either by the simultaneous emission of an electron and an anti-neutrino, or a positron and a neutrino, or by the capture of an atomic electron with the emission of a neutrino. As an example , $^{77}_{34}\mathrm{Ge}$ decays by a series of β-decays to $^{77}_{34}\mathrm{Se}$, here Z increasing by one at each stage:

$$\begin{array}{l} ^{77}_{34}Ge \rightarrow ^{77}_{33} As + e^- + \bar{\nu}_e + 2.75 Mev \\ \qquad\qquad\downarrow \\ \qquad\quad ^{77}_{34}Se + e^- + \bar{\nu}_e + 0.68\, mev. \end{array}$$

$^{77}_{36}\mathrm{Kr}$ decays by emitting a positron and a neutrino. Another sequence of decays ending: in $^{77}_{34}\mathrm{Se}$ is:

$$\begin{array}{l} ^{77}_{36}Kr \rightarrow ^{77}_{35} Br + e^+ + \nu_e + 2.89\,\mathrm{MeV} \\ \qquad\qquad\downarrow \\ \qquad ^{77}_{34}Se + e^+ + \nu_e + 1.36\,\mathrm{MeV}. \end{array}$$

4. γ-DECAY

The gamma γ-decay is the process in which an excited state of a nucleus transforms into a lower-energy state with the difference in energy appearing as electromagnetic radiation, the gamma γ-ray. γ-rays are frequently emitted following α and β-decay and indicate that, just as the fission fragments are generally produced in excited states, both α and β-decay often lead to excited states of their product nuclei as well [12].

When a β-unstable nucleus decays, it may be energetically possible for the transition to be to an excited state of the daughter nucleus. Although the immediate energy release for decay to an excited state is less than that for decay to the ground state, there are many β-decays in which the selection rules make decay to an excited state more likely. The excited state will then itself decay, usually by γ-emission. As an example $^{60}_{27}\mathrm{Co}$ rarely decays directly to the ground state of $^{60}_{28}Ni$, but with 99.9% probability it decays to a state with an excitation energy of 2.50 Mev. The β-emission is quickly followed by the emission of two photons with energies of 1.17 Mev and 1.33 Mev, giving a total γ-energy of 2.50 Mev. In almost all of the remaining 0.1% of β-decays, the electron emission is followed by a single-photon emission of energy 1.33 Mev. In many cases, emission of a number of γ-rays of different discrete energies is observed in radioactive decay, indicating that a number of different excited nuclear states must be produced. Unlike α-decay and β-decay, γ-decay does not produce a transformation of the nucleus to some other nuclide. In the simple case that decay of a nuclide leads only to the ground and first excited states of a daughter, the spectrum of radiations emitted from the daughter nucleus and atom is relatively simple. If the parent decay populates a number of excited states in the daughter, however, the spectrum can become quite complex; each excited state can lead to the emission of γ-rays, conversion electrons, X-rays and Auger electron [12]. In section 5, we give postulate number 4 related to γ and β-decay.

5. MAIN RESULTS

In order to use the Meijer's *G*-functions as the convenient language in nuclear physics, we give our idea in the format of four **postulates** as follows:

1. Each nuclear (or atomic) state can be represented by a Meijer's *G*-function $G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)$ which the values of all orders and parameters determine exactly the one nuclear (or atomic) state.

If each *G*-function specifies a nuclear state then the transitions can be expressed by using properties of G-functions. The equations (2.3), (2.4), (2.5) and (2.6) associate G-functions together.

2. According to (2.3) when the operator z is multiplied by $G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)$ it acts as γ-photon that gives energy to electron to shift from the initial state (lower energy) to the final state (higher energy) which gives another Meijer's *G*-function $G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}+\alpha\\ \mathrm{b_q}+\alpha\end{matrix}\middle| z\right)$.

$$zG^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)=G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}+1\\ \mathrm{b_q}+1\end{matrix}\middle| z\right).$$

Viceversa, if we use the following notation and interpretation:

$$G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}+1\\ \mathrm{b_q}+1\end{matrix}\middle| z\right)=G^{m,n}_{p,q}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_q}\end{matrix}\middle| z\right)z.$$

It acts as $\gamma-$photon that exit from atom because of transition from upper state to lower state. Note that an interaction of γ-rays with the matter is of type "atomic" interaction not "nuclear".

3. According to (2.4) when the operator $z\frac{d}{dz}$ is multiplied by $G_{p,q}^{m,n}\left(\begin{matrix}\mathrm{a_p}\\ \mathrm{b_p}\end{matrix}\middle| z\right)$ it acts as γ-photon that gives energy to electron to shift from the initial state (lower energy) to the final state (high energy) which gives another Meijer's *G*-function . $G_{p+1,q+1}^{m,n+1}\left(\begin{matrix}0,\mathrm{a_p}\\ \mathrm{b_p},1\end{matrix}\middle| z\right)$ Note again that interaction of γ-rays with the matter is of type "atomic" interaction not "nuclear".

4. According to (2.5) when the *G*-function is symmetric in the groups of parameters, then $G_{p,q}^{m,n}\left(\begin{matrix}\mathrm{a}_1,\ldots,a_p\\ \mathrm{b}_1,\ldots,b_{q-1},a_1\end{matrix}\middle| z\right)$ can describe the nuclear excited state before γ or β decay, and $G_{p-1,q-1}^{m,n-1}\left(\begin{matrix}a_2,\ldots,a_p\\ b_1,\ldots,b_{q-1}\end{matrix}\middle| z\right)$ gives lower energy (excited or ground)nuclear state. For example suppose that we want to express beta decays of nuclei $({}^{77}_{32}Ge$ to ${}^{77}_{33}As$ and ${}^{77}_{33}As$ to ${}^{77}_{34}S$ & $({}^{77}_{36}Kr$ to ${}^{77}_{35}Br$ and ${}^{77}_{35}Br$ to ${}^{77}_{34}Se)$ in terms of *G*-functions. Although we do not know which G-functions exactly describe the all nuclei, assume the following example: Using equation (2.5) two times gives

$$G_{4,4}^{2,3} \rightarrow G_{3,3}^{2,2} \rightarrow G_{2,2}^{2,1},$$

and may represent β-decay by emitting electron, while using equation (2.6) two times gives

$$G_{4,4}^{4,1} \rightarrow G_{3,3}^{3,1} \rightarrow G_{2,2}^{2,1},$$

and may represent β-decay by emitting positron.

The postulates above demonstrate the required building blocks for giving newdescribing version of nuclear physics, here for γ-rays, γ-decay and β-decay.

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